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Analytical and numerical study of a nonlinear finance model via dynamical systems approach

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Abstract

Financial systems have become popular inter-disciplinary research topic due to the applicability of various physical laws and dynamical system techniques in their analysis. Limited forecastability and complex dynamical properties of finance systems motivate researchers to use nonlinear dynamical equations to model and analyse such systems. In this paper, a modified mathematical model for finance system has been proposed and studied. The model describes the evolution of the interest rate, the investment demand and the price index in terms of three coupled ordinary differential equations. The equilibrium solutions of the model are determined and their linear stability nature is discussed. Existence of Hopf bifurcation is shown analytically. The time evolution as well as phase diagram of the system are plotted numerically. The bifurcation diagram of the model with respect to relation coefficient (μ) is depicted numerically. It is observed from the bifurcation diagram that the dynamics of the system changes qualitatively with the variation of the relation coefficient (μ). Our analysis shows the presence of steady state, limit cycle, period-2, period-4 and chaotic dynamics in the model with the variation of μ. Therefore the relation coefficient becomes a significant parameter of the proposed model.

Keywords:

Chaos;
Complex financial system;
Stability analysis;
Relation coefficient;
Bifurcation.

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1. Introduction

Dynamical Systems have been used to model ecological systems [1], Physical systems [2], excitable media and many other systems. Chaotic dynamics have been observed in many areas of real life situations. Nonlinear dynamics have recently become more prominent in mainstream economics since unpredictability and instability are typical features of chaotic systems. In finance, stocks and social economics, all types of economic problems are getting more and more complicated, due to the influence of nonlinear factors.

Large-scale, diversity and complexity have manifested themselves in the internal structures of the present economic systems [11]. Sometimes improper choice of parameters in economic model may cause market to operate in an unfitting manner, so the market may become out of control. In this way Financial systems have become popular inter-disciplinary research topic and it has become more and more important to make a systematic and deep study in the internal structural characteristics in such a complicated economic system. Recently Gao and Ma [3] investigated dynamical behavior of finance system models.

In this work we modify the finance system model reported in Gao and Ma [3] by introducing a new relation coefficient $\mu \in (0,1)$ in the model. We have shown that if the parameters in the system are not chosen/ combined judiciously, chaotic phenomena may occur in the system which in turn may force the system

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plunged into stagnant state. We find the equilibrium points of the model and discuss their local stability nature. We also solve the model numerically for different values of μ and the results are discussed.

2. A Mathematical model of Finance system

A dynamic model of finance system composed of three first-order differential equations are reported in Refs. [3–6]. The model describes the time-variation of three state variables: the interest rate *x*, the investment demand, *y*, and the price index, *z*. The factors that influence changes in *x* mainly come from two aspects: first, contradictions from the investment market, i.e., the surplus between investment and savings, and second, structural adjustment from good prices. The changing rate of *y* is in proportion to the rate of investment, and in proportion to an inversion with the cost of investment and interest rates. Changes in *z*, on one hand, are controlled by a contradiction between supply and demand in commercial markets, and on the other hand, are influenced by inflation rates. By choosing an appropriate coordinate system and setting appropriate dimensions for every state variable, Refs. [3–6] offer the simplified finance model as

$$\left.\begin{aligned} \dot{x} &= z + (y-a)x \\ \dot{y} &= 1 - by - x^2 \\ \dot{z} &= -x - cz \end{aligned}\right\} \ldots\ldots\ldots\ldots(1)$$

where $a(\geq 0)$ is the saving amount, $b(\geq 0)$ is the cost per investment, and $c(\geq 0)$ is the elasticity of demand of commercial markets.

Although previous authors [3–6] have considered the simplified model in their papers, it is observed in real world that change of interest rate is not exactly proportional to the change of price index rather it changes with some positive proportion. So wee construct here a slightly modified form of the above model by introducing a new relation coefficient $\mu(>0)$ assuming that the rate of change of interest rate, in real finance system, is directly proportional to some scaled price μz instead of z. The modified form of system (1) is as follows:

$$\left.\begin{aligned} \dot{x} &= \mu z + (y-a)x \\ \dot{y} &= 1 - by - x^2 \\ \dot{z} &= -x - cz \end{aligned}\right\} \ldots\ldots\ldots\ldots(2)$$

3. Analytical Results

To find the equilibrium point of system (2) we solve the system

$$\left.\begin{aligned} \mu z + (y-a)x &= 0 \\ 1 - by - x^2 &= 0 \\ -x - cz &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots(3)$$

The system (2) has the only equilibrium point $\left(0,\ \frac{1}{b},\ 0\right)$ if $c - b\mu - abc \leq 0$ and it has three equilibrium points viz. $\left(\pm\sqrt{\frac{c-b\mu-abc}{c}},\ \frac{\mu+ac}{c},\ \mp\frac{1}{c}\sqrt{\frac{c-b\mu-abc}{c}}\right), \left(0,\ \frac{1}{b},\ 0\right)$ if $c - b\mu - abc \geq 0$.

In the following we study the stability of the system at the equilibrium point $\left(0,\ \frac{1}{b},\ 0\right)$ if $c - b\mu - abc \leq 0$.

Let us make a transformation $X = x,\ \ Y = y - \frac{1}{b},\ \ Z = z$

Then (2) becomes

$$\left.\begin{aligned} \dot{X} &= \left(\frac{1}{b} - a\right)X + \mu Z + XY \\ \dot{Y} &= -bY - X^2 \\ \dot{Z} &= -X - cZ \end{aligned}\right\} \ldots\ldots\ldots\ldots(4)$$

Then the equilibrium point becomes $(0,\ 0,\ 0)$.

The Jacobian (*J*) of system (4) at the equilibrium point $(0,\ 0,\ 0)$ is

$$J = \begin{pmatrix} \frac{1}{b}-1 & 0 & \mu \\ 0 & -b & 0 \\ -1 & 0 & -c \end{pmatrix} \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

The characteristic equation of (5) is

$$(\lambda + b)\left[\lambda^2 + \left(c + a - \frac{1}{b}\right)\lambda + ac - \frac{c}{b} + \mu\right] = 0 \ldots\ldots\ldots\ldots..(6)$$

The eigenvalues of (6) are $\lambda_1 = -b,\ \lambda_2$ and λ_3 are determined by

$$\lambda^2 + \left(c + a - \frac{1}{b}\right)\lambda + ac - \frac{c}{b} + \mu = 0$$

Now $\quad c - b\mu - abc \le 0 \quad \Rightarrow \quad ac - \frac{c}{b} + \mu \ge 0$

The following cases arise:

Case I: $c - b\mu - abc < 0$ and $c + a - \frac{1}{b} > 0$

$\Rightarrow \quad \lambda_2 < 0$ and $\lambda_3 < 0$. Also $\lambda_1 = -b < 0$.

$\therefore$ The equilibrium point $\left(0,\ \frac{1}{b},\ 0\right)$ of the system (2) is the stable convergence.

Case II: $c - b\mu - abc < 0$ and $c + a - \frac{1}{b} < 0$.

$\Rightarrow \quad \lambda_2 > 0$ and $\lambda_3 > 0$. Also $\lambda_1 = -b < 0$.

$\therefore$ The equilibrium point $\left(0,\ \frac{1}{b},\ 0\right)$ of the system (2) is a saddle point.

Case III: $c - b\mu - abc = 0$

$\Rightarrow \quad \lambda_2 = 0$ and $\lambda_3 = -\left(c + a - \frac{1}{b}\right)$.

Subcase I: $0 < c < \sqrt{\mu}$.

$\Rightarrow \quad \lambda_3 = -\left(c + a - \frac{1}{b}\right) = \frac{\mu - c^2}{c} > 0$

$\therefore$ The equilibrium point $\left(0,\ \frac{1}{b},\ 0\right)$ of the system (2) is unstable since one eigenvalue is positive.

Subcase II: $c > \sqrt{\mu}$.

$\Rightarrow \quad \lambda_3 = -\left(c + a - \frac{1}{b}\right) = \frac{\mu - c^2}{c} < 0$

$\therefore$ The equilibrium point $\left(0,\ \frac{1}{b},\ 0\right)$ of the system (2) is nonhyperbolic. The stability nature of this fixed point will be determined numerically.

Case IV: $c - b\mu - abc < 0$ and $c + a - \frac{1}{b} = 0 \quad \Rightarrow c^2 < \mu$.

In general the roots of the characteristic polynomial (6) are

$$\lambda_1 = -b,$$

$$\lambda_2 = \left[-\left(c + a - \frac{1}{b}\right) - \sqrt{\left(c + a - \frac{1}{b}\right)^2 - 4\left(ac - \frac{c}{b} + \mu\right)}\right]/2$$

$$\lambda_3 = \left[-\left(c+a-\frac{1}{b}\right)+\sqrt{\left(c+a-\frac{1}{b}\right)^2-4\left(ac-\frac{c}{b}+\mu\right)}\right]/2$$

Here we consider the parameter a as a bifurcation parameter and a fixed value of a is $a_0 = \frac{1}{b} - c$.

Then the complex conjugate roots can be recast as

$\bar{\lambda}(a) = \alpha(a) \pm i\beta(a)$ where $\alpha(a) = -\left(c+a-\frac{1}{b}\right)$ and $\beta(a) = \sqrt{\left(c+a-\frac{1}{b}\right)^2-4\left(ac-\frac{c}{b}+\mu\right)}$

It follows that $\alpha(a_0) = 0$, $\beta(a_0) \neq 0$

$\therefore$ Under the assumed condition the roots become $\lambda_1 = -b < 0$ and $\lambda_2 = i\sqrt{\mu - c^2}$ and $\lambda_3 = -i\sqrt{\mu - c^2}$ purely imaginary.

Furthermore $\left.\frac{d\alpha}{da}\right|_{a=a_0} = -1 \neq 0$ which is the transversality condition for existence of Hopf Bifurcation.

$\therefore$ At the equilibrium point $\left(0, \frac{1}{b}, 0\right)$ Hopf bifurcation occurs with respect to a.

4. Results of numerical simulation:

The RK4 scheme is used to solve the system

$$\left.\begin{aligned}\dot{x} &= \mu z + (y-a)x \\ \dot{y} &= 1 - by - x^2 \\ \dot{z} &= -x - cz\end{aligned}\right\}$$

numerically using MATLAB. We have kept the parameters a, b, c fixed as far as practicable and varied μ in the range $0 < \mu < 1$ with $(x_0, y_0, z_0) = (0.1, 4.5, -0.1)$. Time evolution of x, y, z and phase diagrams for x-y-z for different cases are shown below.

Case I: $a = 6.0$, $b = 0.1$, $c = 1.0$, $\mu = 0.15$

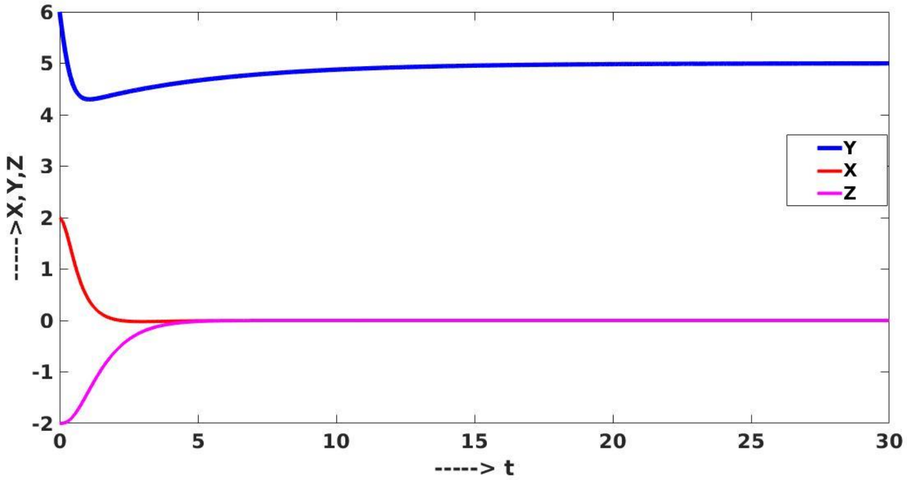

Fig. 1: Time evolution of x, y, z for $a = 6.0$, $b = 0.1$, $c = 1.0$, $\mu = 0.15$

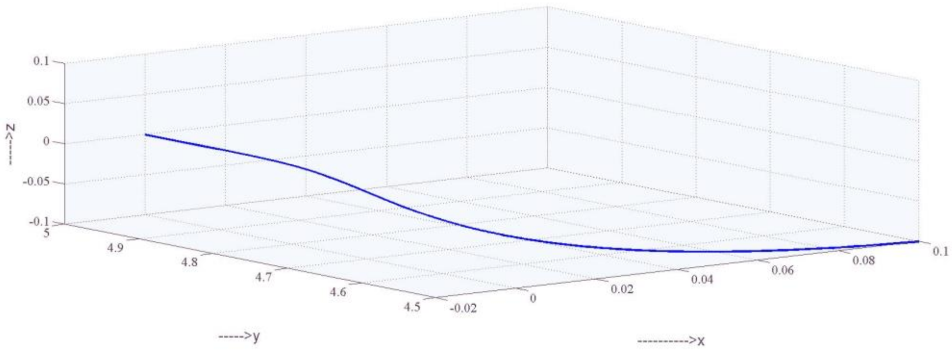

Fig. 2: x-y-z phase diagram for $a = 6.0,\ b = 0.1,\ c = 1.0,\ \mu = 0.15$

Case II: $a = 4.5,\ b = 0.2,\ c = 0.4,\ \mu = 0.3$

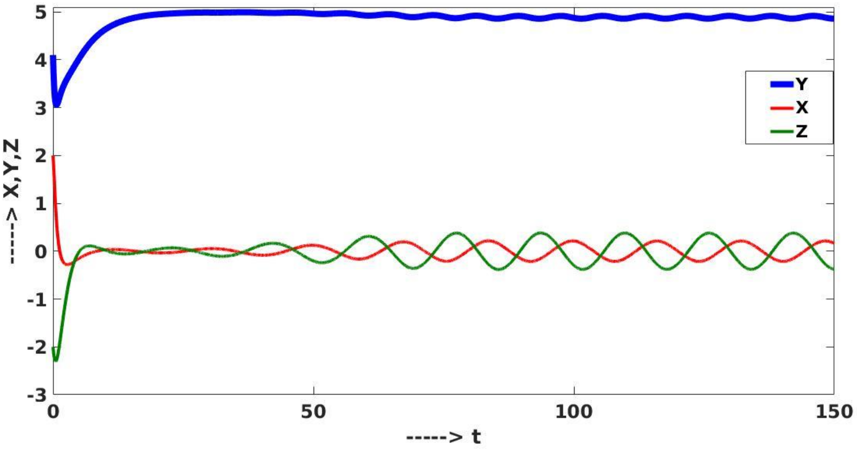

Fig.3: Time evolution of x, y, z for $a = 4.5,\ b = 0.2,\ c = 0.4,\ \mu = 0.3$

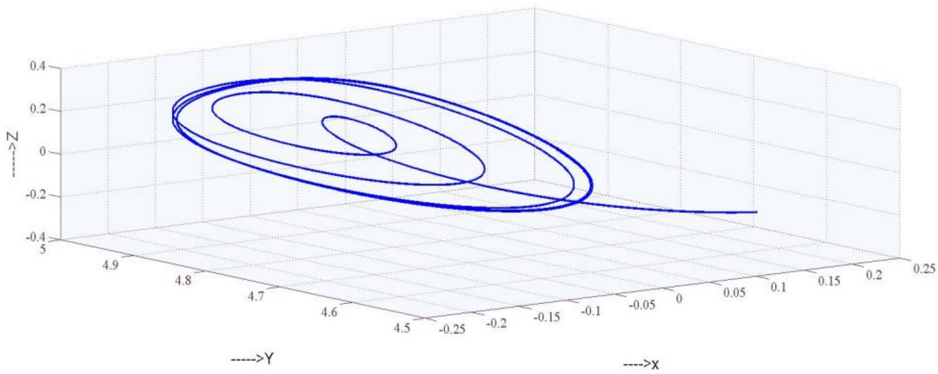

Fig.4: x-y-z phase diagram for $a = 4.5,\ b = 0.2,\ c = 0.4,\ \mu = 0.3$

Case III: Subcase I: $a = 4.5,\ b = 0.2,\ c = 0.125,\ \mu = 0.0625$

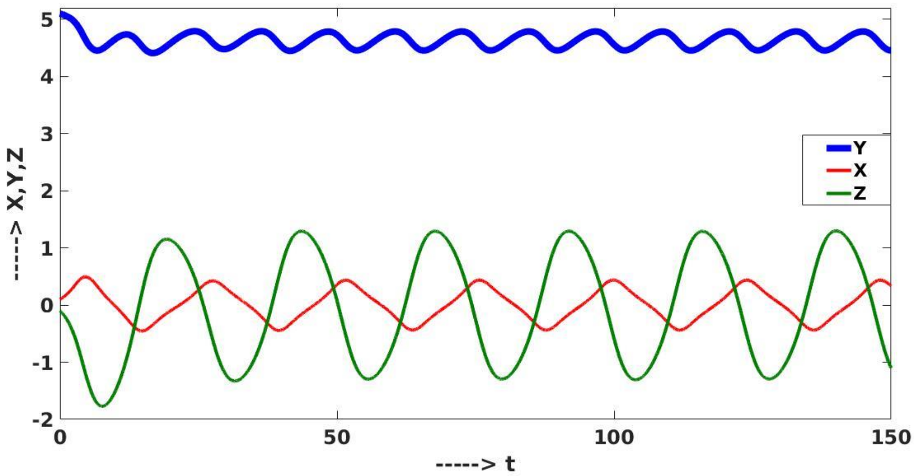

Fig. 5: Time evolution of x, y, z for $a = 4.5,\ \ b = 0.2,\ \ c = 0.125,\ \mu = 0.0625$

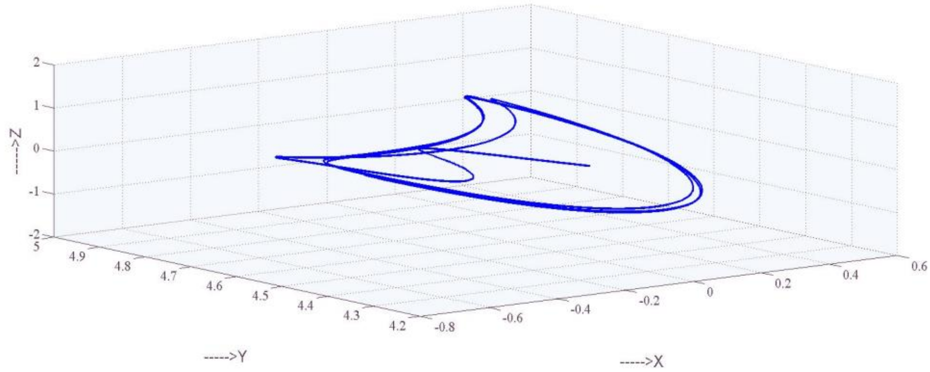

Fig. 6: x-y-z phase diagram for $a = 4.5,\ \ b = 0.2,\ \ c = 0.125,\ \mu = 0.0625$

Subcase II: $a = 4.5,\ \ b = 0.2,\ \ c = 2.0\ \ \mu = \ 1.0$

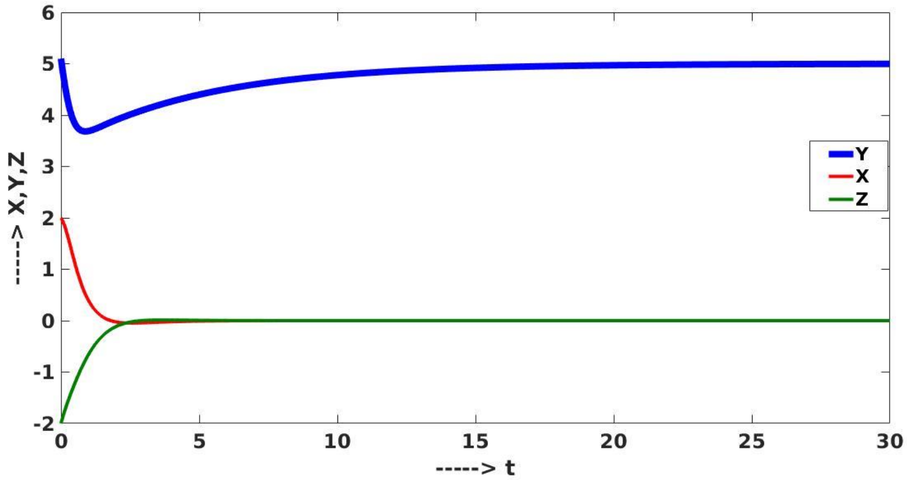

Fig. 7: Time evolution of x, y, z for $a = 4.5,\ \ b = 0.2,\ \ c = 2.0,\ \mu = 1.0$

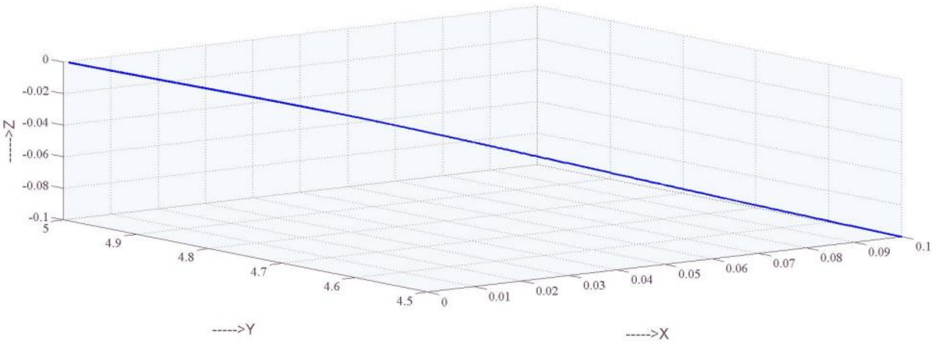

Fig. 8: x-y-z phase diagram for $a = 4.5,\ \ b = 0.2,\ \ c = 2.0,\ \ \mu = 1.0$

Case IV: First we have considered $a = 4.3,\ \ b = 0.2,\ \ c = 0.5,\ \ \mu = \ 0.3$

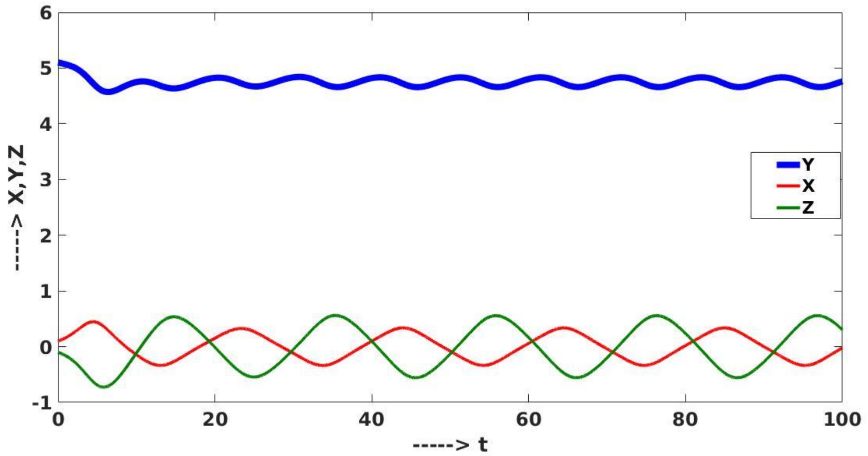

Fig9. : Time evolution of x, y, z for $a = 4.3,\ \ b = 0.2,\ \ c = 0.5,\ \ \mu = 0.3$

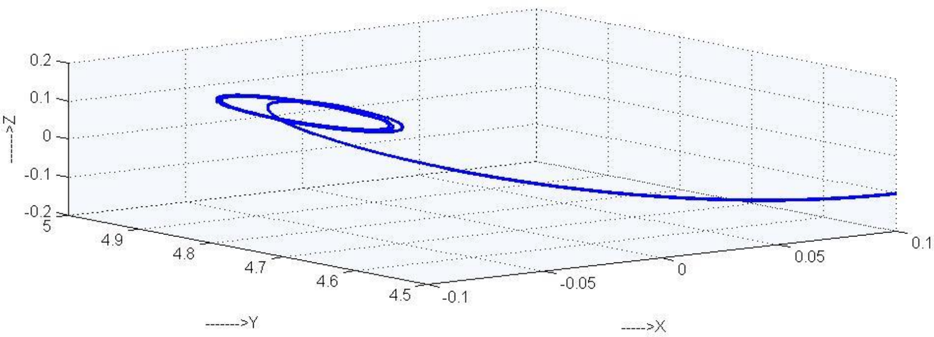

Fig10: x-y-z phase diagram for $a = 4.5,\ \ b = 0.2,\ \ c = 0.5,\ \ \mu = 0.3$

The computational results as shown above in in different cases, are completely consistent with the results of corresponding theory deductions.

The bifurcation diagram of our proposed model (2) with respect to the relation coefficient μ has been done numerically and the bifurcation diagram for x has been presented in Fig. 13. In this case we have chosen the other parameters as $a = 0.6,\ \ b = 0.1,\ \ c = 1.0$. It is clear that the system has steady state behaviour for very low values of μ but it has complex dynamical behaviour for slightly higher values of μ. A slight increment of μ shows oscillatory behaviour of the model. Further increment of μ ultimately makes the dynamics chaotic through the period doubling route. Therefore it is observed that with the variation of μ the dynamics of the system changes significantly and we can conclude that μ is an important parameter of the model.

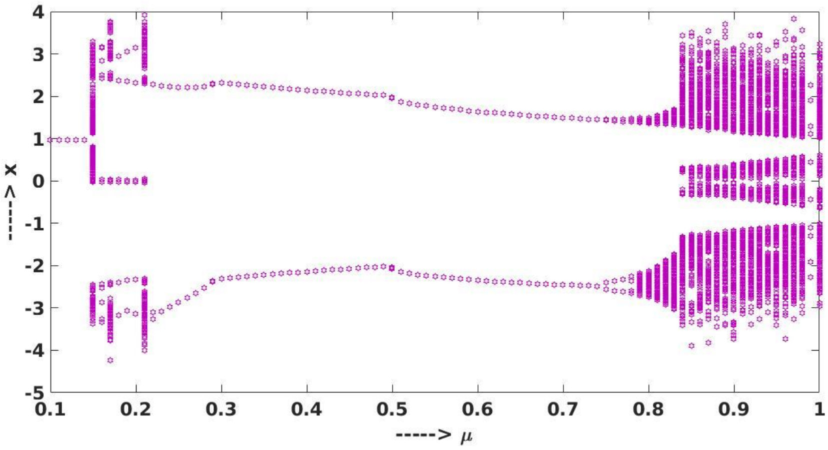

Fig11: bifurcation diagram for x when $a = 0.6,\ b = 0.1,\ c = 1.0$

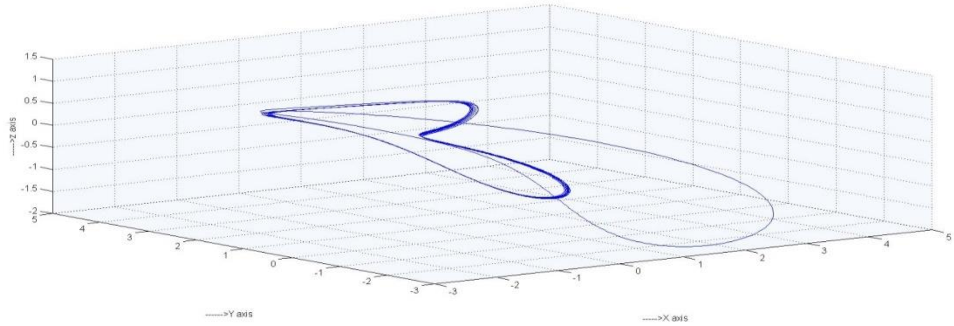

Fig12: x-y-z phase diagram for $a = 0.6,\ b = 0.1,\ c = 1.0,\ \mu = 0.5$

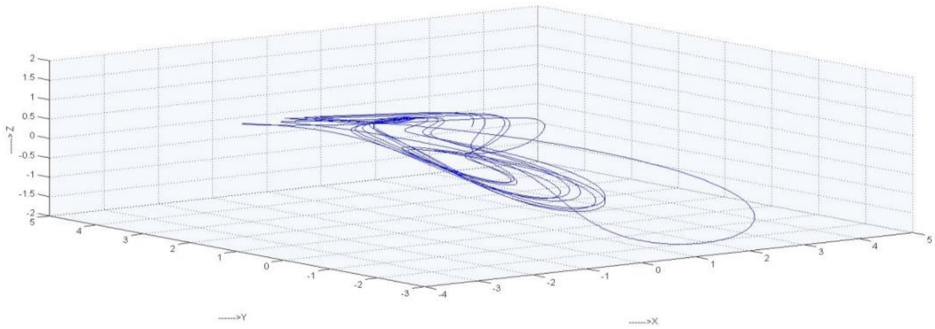

Fig13: x-y-z phase diagram for $a = 0.6,\ b = 0.1,\ c = 1.0,\ \mu = 0.8$

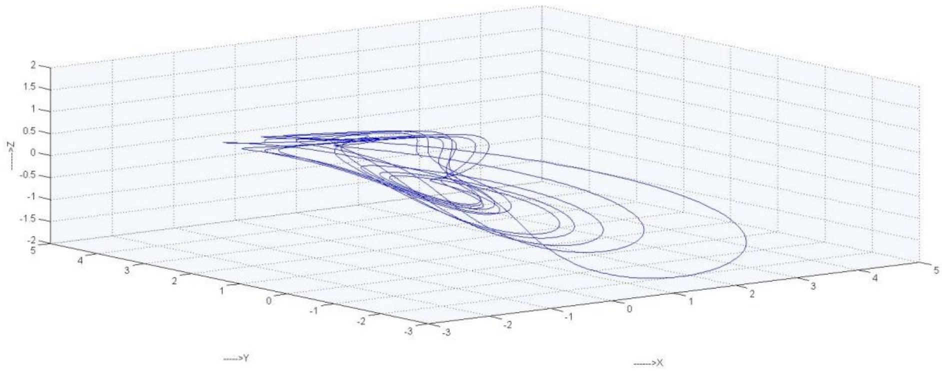

Fig14: x-y-z phase diagram for $a = 0.6,\ b = 0.1,\ c = 1.0,\ \mu = 0.99$

5. Conclusion

From the theoretical discussion and Numerical simulation of our proposed model we have noticed the existence of chaos, period-4 orbit, period-2 orbit, limit cycle and steady state behavior of the system with the variation of μ in the range $0 < \mu < 1$ keeping the other parameters fixed. Our analysis shows the sensitive dependence of the dynamical behavior of the finance system on the newly introduced relation coefficient μ.

Although the saving amount, the cost per-investment and the demand elasticity of commercials are interdepending on each other in our proposed model, we have shown that the presence of relation coefficient μ is significant. The parameter μ may assume different values along with our model selected values. Accordingly, the dynamical characteristic of the system may be different depending on values of μ.

We have also shown through our study that the rational control of μ may become a prerequisite and guarantee avoiding the occurrence of chaos in the financial system. This may develop further scope of research regarding the pattern of dependence among the state variables.

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